Minimum-Fuel Rendezvous Techniques

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Nomenclature

C	=	the velocity requirement number defined by (8)
M	=	the initial mass of S
S	=	a space vehicle intended to rendezvous with T
T	=	a target moving in a circular orbit
i_T, j_T	=	the unit vector in the direction of r_T, the target velocity resp.
k	=	the maximum acceleration of the engine of S divided by $\|r_T\|\omega_T^2$
m, n	=	integers
n_T	=	the unit vector completing the right-handed orthogonal set with i_T and j_T
r_S, r_T	=	the vectors from the center of the gravitational field to S, T
r	=	$r_S - r_T$
v_T	=	the velocity of T
v_e	=	the exhaust velocity of the engine
α, β, γ	=	the driving acceleration components of S in the system i_T, j_T, n_T divided by $\|r_T\|\omega_T^2$
ξ, η, ζ	=	the components of r in the system i_T, j_T, n_T divided by $\|r_T\|$
$\xi_1, \xi_2, \xi_3, \xi_4$	=	canonical variables defined by (3)
ω_T	=	the angular rate of T
τ	=	dimensionless time $= -\omega_T t$
τ_{pl}, τ_{sp}	=	the maneuver time; plane part of motion, spatial part of motion
$\tau_0, \tau_1, \tau_2, \tau_3$	=	location of firing intervals
$\varphi_0, \varphi_1, \varphi_2, \varphi_3$	=	duration of firing intervals
δ	=	an auxiliary quantity defined by (13)
ν	=	integer
σ_2	=	an auxiliary angle defined by (17)
ψ	=	an auxiliary angle defined by (13)
ΔM	=	the mass expenditure of S during the maneuver
$(')$	=	differentiation with respect to τ

Introduction

THIS note is devoted to the study of the rendezvous between a space vehicle S and a target T moving in a circular orbit. The relative motion is described by the Clohessy-Wiltshire equations.[1] Minimum-fuel thrust programs can be determined by way of the maximum principle of Pontrjagin.[2] It gives information on the structure of optimum thrust functions, but unfortunately, does not give explicit information on the correspondence between the initial constellation of the bodies and the control parameters.[3,4] We overcome the difficulty involved and advance the discussion by a new analysis of the relevant equations. This analysis seems to lie very close to the physical nature of the problem and leads to explicit solutions. The set of basic equations divides into two subsets, one describing the motion in the target plane, and the other the motion orthogonal to the target plane. Minimum-fuel controls for the second part of motion are explicitly given. The investigation of the plane part of motion proceeds by three stages.[5] First, we introduce canonical variables. They have an immediate physical significance, namely the energy difference between the bodies, the angular difference between them, and two variables indicative of the difference between actual velocity of S and the circular orbit velocity determined by the actual position. Second, we show that no thrust is required in the direction of the target radius vector. The system remains completely controllable,[6,7] and we have the advantage of operating in one-dimensional control space. Third, we solve the resulting simple system explicitly by a control function, which shows the essential features of minimum-fuel rendezvous. By suitable choice of maneuver time, the difference of actual fuel consumption and minimum consumption determined by the energy difference becomes arbitrarily small. One important condition for the existence of such controls is the no-intersection property of the initial constellation which is explained in the text.

Description of Motion

Let us consider the rendezvous between S and T in an inverse-square central gravitational field, T moving in a circular orbit. The well-known Clohessy-Wiltshire equations describing the relative motion are as follows:

$$\left.\begin{aligned} \xi'' - 3\xi + 2\eta' &= \alpha \\ -2\xi' + \eta'' &= \beta \end{aligned}\right\} \tag{1}$$

$$\zeta'' + \zeta = \gamma \tag{2}$$

They are valid under the assumption of a small distance between the two objects compared with $|r_T|$. Normalization, as introduced in (1) and (2), affords independence of a special target orbit. A basic theoretical study deals with the rendezvous maneuver as the following open-loop guidance problem: to determine accelerations to be produced by engines as functions of time and submitted to certain optimization criteria, so that a given initial state ξ, ξ', η, η'; τ, $\zeta' \neq 0$ is transferred to the zero state.

Equations (1) and (2) show that there is no cross-coupling between the part of motion in the target plane represented by (1) and the part of motion orthogonal to the target plane represented by (2). In so far as the linearization implied is feasible, one may therefore handle both parts of motion separately, in mathematical treatment as well as in actual performance. We may provide different durations for each component of motion.

The canonical structure of the spatial part of motion is clear from (2). For the plane part of motion, we introduce the (real) canonical variables as follows:

$$\begin{aligned} \xi_1 &= \eta' - 2\xi & \xi_2 &= -\tfrac{1}{3}(\eta + 2\xi') \\ \xi_3 &= \eta' - \tfrac{3}{2}\xi & \xi_4 &= -\tfrac{1}{2}\xi' \end{aligned} \tag{3}$$

The corresponding system of linear differential equations is as follows:

$$\begin{pmatrix} \xi_1' \\ \xi_2' \\ \xi_3' \\ \xi_4' \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -\frac{2}{3} & 0 \\ 0 & 1 \\ -\frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{4}$$

The new state variables have a direct physical meaning. ξ_1 is the linear term of the kinematic energy difference between the two objects, divided by $|v_T|^2$. If coasting motion of the body S is assumed, ξ_3 and ξ_4 execute a harmonic oscillation. From

$$|r_S| = |r_T|(1 + \xi) \tag{5}$$

it follows that $\xi_4 = 0$ indicates passage of S through its apogee or perigee. Hence, the amplitude $(\xi_3^2 + \xi_4^2)^{1/2}$ is an indicator of the deviation of the S orbit from a circular orbit. It is easy to show from (3) that if this amplitude 1) is less than $|\xi_1|$, the two orbits have no points in common; 2) equals $|\xi_1|$, the two orbits have a common tangent; or 3) exceeds $|\xi_1|$, the two orbits intersect. Condition 1 is called the no-intersection property. ξ_2 is a generalized phase between the

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two objects. For example, the so-called Hohmann phase, the initial constellation to which the well-known Hohmann impulses apply, is defined by $\xi_2 = \pi\xi_1/2$.

The acceleration program can be simplified by preprograming the thrust direction as a function of time, independent of the state of the system.[7] Our system (4) remains completely controllable[6] if only β accelerations are applied. That means we can transfer each given initial state to the zero state by only producing accelerations in the direction orthogonal to the target radius vector. The degree of mathematical and technical simplification of the maneuver is obvious. It will be demonstrated later that there is also no loss of performance in terms of fuel consumption. Solution of (4) assuming $\alpha = 0$ and introducing the rendezvous boundary conditions yields the formulas

$$\left.\begin{aligned} \xi_1 &= \int_0^{\tau_{\rm pl}} \beta(\tau_{\rm pl} - \tau)d\tau \qquad \xi_2 = \int_0^{\tau_{\rm pl}} \tau\beta(\tau_{\rm pl} - \tau)d\tau \\ \xi_3 &= \int_0^{\tau_{\rm pl}} \cos\tau\beta(\tau_{\rm pl} - \tau)d\tau \\ \xi_4 &= \int_0^{\tau_{\rm pl}} \sin\tau\beta(\tau_{\rm pl} - \tau)d\tau \end{aligned}\right\} \qquad (6)$$

The left sides denote the initial state, the arguments having been omitted.

Attainable Fuel Minima

One of the most important performance criteria is "low fuel consumption." The ratio of mass expenditure and initial mass is as follows:

$$\Delta M/M = 1 - \exp\{-C|v_T|/v_c\} \qquad (7)$$

using the term "velocity requirement number" defined by

$$C = \int_0^{\tau_{\rm pl}} (\alpha^2 + \beta^2)^{1/2}d\tau + \int_0 |\gamma|d\tau = C_{\rm pl} + C_{\rm sp} \qquad (8)$$

$C_{\rm pl}$ and $C_{\rm sp}$ should be as small as possible. We consider first the plane part of motion and conclude from (4) and (8) that the inequality

$$C_{\rm pl} = \int_0^{\tau_{\rm pl}} (\alpha^2 + \beta^2)^{1/2}d\tau \geqslant \int_0^{\tau_{\rm pl}} |\beta|d\tau \geq \left|\int_0^{\tau_{\rm pl}} \beta d\tau\right| = |\xi_1| \qquad (9)$$

holds. Equation (9) shows the significance of $|\xi_1|$ as the absolute minimum of the coplanar velocity requirement number. This minimum is attained if and only if no α-acceleration is produced, and β-acceleration is without inversion of sign. In what follows we assume $\alpha = 0$. The third and fourth of Eqs. (6) lead to the inequality

$$C_{\rm pl} = \int_0^{\tau_{\rm pl}} |\beta|d\tau \geq \left|\int_0^{\tau_{\rm pl}} \beta e^{i\tau}d\tau\right| = |\xi_3 + i\xi_4| = (\xi_3^2 + \xi_4^2)^{1/2} \qquad (10)$$

Equation (9) combined with (10) yields

$$C_{\rm pl} \geq \max\{|\xi_1|, (\xi_3^2 + \xi_4^2)^{1/2}\} \qquad (11)$$

To achieve a velocity requirement number near the absolute optimum, therefore, the condition $(\xi_3^2 + \xi_4^2)^{1/2} \leq |\xi_1|$ is necessary. We will prove later that the following basic and important inverse is valid. Suppose thrust capability is high enough, and let the given initial constellation satisfy $(\xi_3^2 + \xi_4^2)^{1/2} < |\xi_1|$. Then a control schedule with $\alpha = 0$ and proper maneuver time can be determined, the actual velocity requirement number of which differs from the absolute minimum $|\xi_1|$ by an arbitrarily small amount. Hence, we are justified in considering only β-acceleration.

Concerning the spatial part of motion, it can be easily seen from (2) that

$$C_{\rm sp} \geq (\zeta^2 + \zeta'^2)^{1/2} \qquad (12)$$

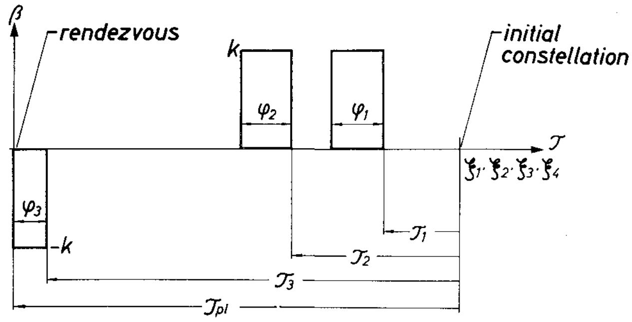

Fig. 1 Control program, plane component.

holds, where $(\zeta^2 + \zeta'^2)^{1/2}$ is the angle between the planes of S and T.

Explicit Control Schedules

Equations (6) admit a very simple closed solution. Let us consider a control program as shown in Fig. 1. Two firing intervals with positive maximum acceleration k are followed by a firing interval with negative maximum acceleration $-k$. A proper coasting time at the beginning is provided. For the sake of simplicity we assume $\xi_1 > 0$. Let δ, ψ be defined by

$$\delta = \pi/2 - \xi_2/\xi_1 \qquad \sin\psi = \xi_3/(\xi_3^2 + \xi_4^2)^{1/2}$$
$$\cos\psi = \xi_4/(\xi_3^2 + \xi_4^2)^{1/2} \qquad (13)$$

and let duration and location of the firing intervals of the control schedule be given by

$$\varphi_1 = \varphi_2 + \varphi_3 \qquad \varphi_2 = \xi_1/2k$$
$$\varphi_3 = \varphi_2(\nu\pi - \psi + \delta)/\psi n\pi \qquad (14)$$

$$\left.\begin{aligned} \tau_1 &= \nu\pi - \psi + (-1)^\nu \frac{|\sigma_2|}{2} - \varphi_2/2 - \varphi_3 \\ \tau_2 &= (\nu + 1)\pi - \psi + (-1)^{\nu+1} \frac{|\sigma_2|}{2} - \varphi_2/2 \\ \tau_3 &= 2n\pi + \tau_1 \end{aligned}\right\} \qquad (15)$$

$$\sin\frac{|\sigma_2|}{2} = \frac{(\xi_3^2 + \xi_4^2)^{1/2}(\varphi_2/2)}{\xi_1 \sin(\varphi_2/2)} \qquad (16)$$

$n \geq 1$ is an arbitrary integer that determines the total time of maneuver; ν is an integer, which should be chosen so that, first $\varphi_3 \geq 0_1\tau_1 \geq 0$, second, no overlapping of firing intervals occurs, and third, φ_3 is as small as possible. Substitution of (14), (15) into (6) shows that we have actually stated a solution of (6). The resulting velocity requirement number is given by

$$C_{\rm pl} = k(\varphi_1 + \varphi_2 + \varphi_3) = \xi_1[1 + (\nu\pi - \psi + \delta)/n\pi] \qquad (17)$$

This formula supplies the proof deferred from the preceding section. By choosing n sufficiently large, the difference $C_{\rm pl} - \xi_1$ can be made arbitrarily small. Equation (16) shows that our solution exists, if the no-intersection property holds, and if k is large enough so that the right side of (16) is $\leqslant 1$.

For the important special case $\xi_3 = \xi_4 = 0$, two subcases have to be considered. Let the generalized Hohmann phase be defined by $\delta + \varphi_2/2 = 0$. 1) Suppose $\delta + \varphi_2/2 \leqslant 0$, which means that S lags behind the Hohmann phase. One obtains

$$\varphi_3 = 0 \qquad \tau_1 = -\delta - \varphi_2/2 \qquad \tau_2 = \pi + \tau_1 \qquad (18)$$

The other data remain unchanged. In that case the maneuver is already accomplished in time $\tau_2 + \varphi_2$ with $C_{\rm pl} = \xi_1$. 2) Suppose $\delta + \varphi_2/2 > 0$, which means that S leads the

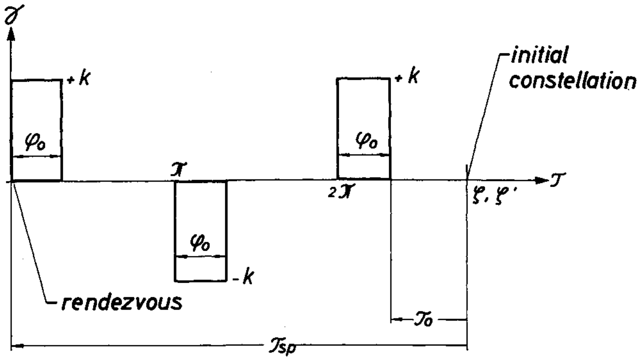

Fig. 2 Control program, spatial component.

Hohmann phase. Now one obtains

$$\varphi_3 = \varphi_2(\delta + \varphi_2/2)/(n\pi - \varphi_2)$$
$$\tau_1 = 0 \qquad \tau_2 = n\pi + \varphi_3 \tag{19}$$

The other data remain unchanged again. The velocity requirement number is greater than the optimum value and increases, for fixed n, with increasing phase difference between S and T. The difference between C_{p1} and ξ_1, however, may be adequately compensated by choice of suitable duration of the maneuver. We have achieved the optimum in the sense of Pontrjagin. The Pontrjagin switching function for the limit case $k \to \infty$ (impulses) is given in closed form by

$$\chi(\tau) = -1 + (1 - \cos\tau)/2n + (\tau + \sin\tau)/n\pi \tag{20}$$

Equation (2) describing the spatial component of motion also admits a simple closed solution. In Fig. 2 the control is demonstrated. We provide firing intervals all of a fixed duration φ_0, with maximum magnitude of acceleration and alternating sign. These intervals are separated by coasting periods of duration $\pi - \varphi_0$. At the beginning a proper coasting time τ_0 is located. It is easily confirmed by (2) that

$$tg(\tau_0 + \varphi_0/2) = \xi/\xi' \qquad \sin(\varphi_0/2) = (\zeta^2 + \zeta'^2)^{1/2}/2km \tag{21}$$

holds; m has to be chosen large enough, so that φ_0 can be determined by (21). The resulting velocity requirement number is given by

$$C_{sp} = km\varphi_0 = (\zeta^2 + \zeta'^2)^{1/2}(\varphi_0/2)/\sin(\varphi_0/2) \tag{22}$$

This formula shows that C_{sp} tends to the absolute optimum $(\zeta^2 + \zeta'^2)^{1/2}$, if m tends to infinity. We have achieved the optimum in the sense of Pontrjagin. The corresponding switching function is as follows:

$$\chi(\tau) = \cos(\tau - \varphi_0/2)/\cos(\varphi_0/2) \tag{23}$$

Concluding Remarks

The following statements must be taken into account for a low-fuel rendezvous maneuver: 1) the initial orbit of the space vehicle should lie within the target orbit; 2) there should be an angular difference between the two bodies such that the space vehicle lags behind the Hohmann phase; and 3) deviations from 2) can be compensated by a sufficient maneuver time. Numerical solutions to most optimization problems can be obtained by the well-known methods of the calculus of variations, but they rarely provide theoretical insight. They are indispensable, of course, but science is asked to find general properties as the backbone of computer solutions. This note has answered such questions for the case of rendezvous to a circular moving target.

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Stresses in an Ablating, Viscoelastic, Case-Bonded Cylinder

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SIGNIFICANT strains during pressurization of a ported propellant grain will usually occur near star points (or near the inside radius of a smooth bore grain). Viscoelastic behavior of the propellant and elastic behavior of the motor case will control the magnitude of these strains. Pressurization rate and burning rate will dictate the rise of strain at a particular location and the time at which that location disappears. As a first step toward calculating grain behavior for a realistic burning history, it was decided to confine the analysis to an idealized geometry, e.g., a case-bonded cylinder under plane strain or plane stress. As indicated later, an elastic solution, which takes into account the actual end conditions (and appropriate stress concentration factors), should provide a sufficiently accurate correction to the solution for the ideal geometry. Thus, it was postulated that the important quality of the analysis was to be the retention of realistic conditions of pressurization and ablation. In this light, other simplifications, such as linear viscoelastic behavior, small deformations, and propellant incompressibility and isotropy are assumed.

Formulation

For an orthogonal, curvilinear coordinate system, propellant response is formulated by the Boltzmann superposition integral

$$e_{ij} = \frac{1}{2}\int_0^{t_n} \Phi(t_n - t)\frac{\partial}{\partial t}S_{ij}dt \tag{1}$$

where e_{ij} and S_{ij} are the deviatoric components of the strain and stress tensors, and $\Phi(t)$ is a creep compliance function for shear

$$e_{ij} = \epsilon_{ij} - \tfrac{1}{3}\delta_{ij}\theta \qquad S_{ij} = \sigma_{ij} - \tfrac{1}{3}\delta_{ij}\Theta$$

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